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A Note on Real vs Complex Best Chebyshev Approximation on an Interval

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Let $E_{nn}^{\mathrm{R}}(f)$ ($E_{nn}^{\mathrm{C}}(f)$) be the error in the best Chebyshev approximation of a real continuous function f on $[-1, 1]$ by real (complex) rational functions of type (n, n). We show that the ratio $E_{nn}^{\mathrm{C}}(f)/E_{nn}^{\mathrm{R}}(f)$ may be arbitrarily close to $\frac{1}{2}$ and that for the class of even functions and $n=1$ this bound is sharp. We also prove that $\inf\{E_{11}^{\mathrm{C}}(f)/E_{11}^{\mathrm{R}}(f): E_{11}^{\mathrm{R}}(f)>0\}$ is positive.

For any pair (m, n) of non-negative integers let π_{mn}^{R} and π_{mn}^{C} denote the sets of rational functions of type (m, n) with real and complex coefficients, respectively. For any continuous real function f on $[-1, 1]$ we set

$$E_{mn}^{\mathrm{R}}(f) = \inf_{r \in \pi_{mn}^{\mathrm{R}}} \|f - r\|, \qquad E_{mn}^{\mathrm{C}}(f) = \inf_{r \in \pi_{mn}^{\mathrm{C}}} \|f - r\|,$$

where $\|\phi\|$ denotes the supremum norm of ϕ on $[-1, 1]$.

Many authors investigated the phenomenon $E_{mn}^{\mathrm{C}}(f) < E_{mn}^{\mathrm{R}}(f)$, which can occur for a real function f (see Varga [5, Chap. 5] and Trefethen and Gutknecht [4] for the history of this question and for further references). In particular, many efforts have been made to determine the value of

$$\gamma_{mn} = \inf\{E_{mn}^{\mathrm{C}}(f)/E_{mn}^{\mathrm{R}}(f): f \in C[-1, 1] \backslash \pi_{mn}^{\mathrm{R}}\}. \tag{1}$$

In 1982 Ellacott [2] proved that if p is a polynomial of degree $m+1$, then $E_{mn}^{\mathrm{C}}(p)/E_{mn}^{\mathrm{R}}(p) \geq \frac{1}{2}$, provided $m \geq n$. This result suggested the question whether $\frac{1}{2}$ is actually a lower bound for γ_{mn} $(m \geq n)$, and, if so, whether it is sharp.

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Recently Trefethen and Gutknecht [4] proved that $\gamma_{mn} = 0$, provided $m \leqslant n-3$. They also showed that Ellacott's result holds for $n \leqslant 2m+1$; and so, it holds even when $\gamma_{mn} = 0$.

In this note we present some partial results, which suggest that $\frac{1}{2}$ might be the right bound at least for the case $m = n$, with the infimum in (1) being restricted to somewhat smaller (but still wide) class of functions.

We start with the following example.

EXAMPLE. Consider the function

$$f(x) = \frac{1}{2}\left(\frac{x-\alpha i}{x+\alpha i}\right)^n + \frac{1}{2}\left(\frac{x+\alpha i}{x-\alpha i}\right)^n \qquad \alpha > 0. \tag{2}$$

Clearly, f is continuous and real on $[-1, 1]$. By choosing

$$r^*(x) = \frac{1}{2}\left(\frac{x-\alpha i}{x+\alpha i}\right)^n \in \pi_{nn}^{\mathrm{C}},$$

we obtain $\|f - r^*\| = \frac{1}{2}$. It follows that

$$E_{nn}^{\mathrm{C}}(f) \leqslant \tfrac{1}{2}. \tag{3}$$

We now turn to estimating $E_{nn}^{\mathrm{R}}(f)$ from below. The function $x \to (x-\alpha i)/(x+\alpha i)$ maps $(-\infty, \infty)$ bijectively onto $\{z\colon |z| = 1\} \setminus \{1\}$. Therefore, as x increases from $-\infty$ to $+\infty$, $((x-\alpha i)/(x+\alpha i))^n$ traverses the circle $|z| = 1$ n times, omitting the point 1 once. Hence there exist $2n-1$ points $x_1 < x_2 < \cdots < x_{2n-1}$ such that

$$\left(\frac{x_k-\alpha i}{x_k+\alpha i}\right)^n = (-1)^k, \qquad k = 1,..., 2n-1. \tag{4}$$

Straightforward calculation gives the values of x_k:

$$x_k = -\alpha \cot \frac{\pi k}{2n}, \qquad k = 1,..., 2n-1.$$

If $\alpha > 0$ is small enough, then $x_k - s$ lie in the interval $(-1, 1)$, and we obtain, from (4) and (2):

$$f(x_k) = (-1)^k, \qquad k = 1,..., 2n-1.$$

At the points ± 1, f attains the same value, which for α small is close to 1:

$$f(-1) = f(1) = 1 - O(\alpha).$$

We have thus found $2n+1$ consecutive points of $[-1, 1]$:

$$-1 = x_0 < x_1 < \cdots < x_{2n-1} < x_{2n} = 1$$

such that $f(x_k)f(x_{k+1}) < 0$, $k = 0, 1,..., 2n-1$.

Applying de la Vallée Poussin theorem [3], we deduce that $E^{\mathrm{R}}_{2n-1,2n-1}(f) \geqslant \min\{|f(x_k)|: k=0,..., 2n\} \geqslant 1-O(\alpha)$, which of course implies

$$E^{\mathrm{R}}_{nn}(f) \geqslant 1 - O(\alpha). \tag{5}$$

The estimations (3), (5) yield for $\alpha > 0$ small enough:

$$\frac{E^{\mathrm{C}}_{nn}(f)}{E^{\mathrm{R}}_{nn}(f)} \leqslant \frac{1}{2} + O(\alpha). \tag{6}$$

It follows that for any $n \geqslant 1$ there exists a function f for which the ratio (6) is arbitrarily close to $\frac{1}{2}$, and we obtain

THEOREM 1. *For any* $n \geqslant 1$, $\gamma_{nn} \leqslant \frac{1}{2}$.

For the case $n = 1$ Bennett *et al.* [1] proved that if f is even and satisfies $0 = f(0) \leqslant f(x) \leqslant f(1) = 1$ on $[0, 1]$ then

$$\tfrac{1}{4} < E^{\mathrm{C}}_{11}(f) \leqslant E^{\mathrm{R}}_{11}(f) \leqslant \tfrac{1}{2}, \tag{7}$$

which implies that for any such function

$$\frac{E^{\mathrm{C}}_{11}(f)}{E^{\mathrm{R}}_{11}(f)} > \frac{1}{2}. \tag{8}$$

The argument they used to prove (7) can actually be applied to any continuous real function f that satisfies

(i) $-M \leqslant f(x) \leqslant M (M > 0)$ on $[-1, 1]$;

(ii) there exist three points $-1 \leqslant x_1 < x_2 < x_3 \leqslant 1$ such that $f(x_k) = \lambda(-1)^k M$ for $k = 1, 2, 3$ (with $\lambda = 1$ or $\lambda = -1$).

Consequently, for any such function it holds that

$$\tfrac{1}{2}M < E^{\mathrm{C}}_{11}(f) \leqslant E^{\mathrm{R}}_{11}(f) \leqslant M, \tag{7'}$$

which yields (8).

With this observation in mind it is easy to establish the following result:

PROPOSITION. *Let* $f \in C[-1, 1] \backslash \pi_{11}^{\mathbf{R}}$ *and let* e_{10}^* *be the best approximant to* f *from* $\pi_{10}^{\mathbf{R}}$. *Then,*

$$\frac{E_{11}^{\mathbf{C}}(f-e_{10}^*)}{E_{11}^{\mathbf{R}}(f-e_{10}^*)} > \frac{1}{2}.$$

Proof. Note that the function $f-e_{10}^*$ satisfies the conditions (i), (ii) above (cf. [3, p. 161]) with $M = \|f-e_{10}^*\|$. ∎

If the polynomial e_{10}^* is constant, that is, if $E_{10}^{\mathbf{R}}(f) = E_{00}^{\mathbf{R}}(f)$ (this holds in particular for any even function), then $E_{11}(f-e_{10}^*) = E_{11}(f)$, and, again, we obtain (8). If we note that the function f of the Example above was even, we deduce:

THEOREM 2.

$$\inf\{E_{11}^{\mathbf{C}}(f)/E_{11}^{\mathbf{R}}(f) : f \in C[-1, 1] \backslash \pi_{11}^{\mathbf{R}} \text{ s.t. } E_{10}^{\mathbf{R}}(f) = E_{00}^{\mathbf{R}}(f)\} = \tfrac{1}{2}.$$

Although it is not clear whether $\gamma_{11} = \frac{1}{2}$ or not, one thing can be asserted.

THEOREM 3. $\gamma_{11} > 0$.

Proof. Trefethen and Gutknecht [4] proved that $\gamma_{01} > 0$. As it stands, Theorem 3 follows from their result. Indeed, assume that $\gamma_{11} = 0$. Then given $\varepsilon > 0$ one can find $f \in C[-1, 1]$ and $c \in \pi_{11}^{\mathbf{C}}$ such that

$$\|f-c\| < \varepsilon \qquad \text{and} \qquad E_{11}^{\mathbf{R}}(f) = 1 \tag{9}$$

From (9) follows that $\|\text{Im}\, c\| < \varepsilon$. Hence there exists $\delta > 0$ such that $|\text{Im}\, c(1+\delta)| < \varepsilon$. We observe now that the transformation $\phi: t \to ((1+\delta)t+1)/(t+(1+\delta))$ maps $[-1, 1]$ bijectively onto itself and that the change of the argument x by $\phi(t)$ preserves the classes $C[-1, 1]$, $\pi_{11}^{\mathbf{R}}$ and $\pi_{11}^{\mathbf{C}}$ and preserves norms. Hence, the functions $\tilde{f} = f \circ \phi$ and $\tilde{c} = c \circ \phi$ satisfy

$$\|\tilde{f}-\tilde{c}\| < \varepsilon \qquad \text{and} \qquad E_{11}^{\mathbf{R}}(\tilde{f}) = 1. \tag{9'}$$

In view of the choice of δ, we also obtain

$$|\text{Im}\, \tilde{c}(\infty)| = |\text{Im}\, c(\phi(\infty))| = |\text{Im}\, c(1+\delta)| < \varepsilon. \tag{10}$$

Define now $g(t) = \tilde{f}(t) - \text{Re}\, \tilde{c}(\infty)$. Then

$$E_{01}^{\mathbf{R}}(g) \geqslant E_{11}^{\mathbf{R}}(g) = E_{11}^{\mathbf{R}}(\tilde{f}) = 1 \tag{11}$$

by (9′), and

$$\|g-(\tilde{c}-\tilde{c}(\infty))\|=\|\tilde{f}-\tilde{c}+i\,\mathrm{Im}\,\tilde{c}(\infty)\|<2\varepsilon \tag{12}$$

by (9′), (10).

Since $\tilde{c}-\tilde{c}(\infty)\in\pi_{01}^{\mathrm{C}}$, we obtain from (12) that $E_{01}^{\mathrm{C}}(g)<2\varepsilon$, which together with (11) implies $\gamma_{01}<2\varepsilon$. It follows that $\gamma_{01}=0$, contradicting the above-mentioned result of Trefethen and Gutknecht. ∎

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